An Equilibrium Concept for Extensive-Form Games Played by Boundedly Rational Agents[*]

Zafer Akin[†‡]

Very Preliminary

Abstract

This paper studies an equilibrium concept for both one and two player extensive form games. The equilibrium concept, by construction, embraces different bounded rationality approaches in one framework. A special case that examines the games played by potentially time-inconsistent agents is presented.

1 Introduction

Consider an agent who has the choices of going home directly after work or going to a bar with friends and having a drink before going home. He may have different thoughts such as "if I go to a bar and have a drink, I may end up getting drunk which I do not want" or "there is no harm in having just one drink, then I can go home, no problem!" With these thoughts, the agent may end up directly going home or getting drunk or going home after one drink. It really depends on what kind of state of mind the agent has. What should this person do? Given his state of mind at 5.30pm (end of the day at work), he will compare the options he has by taking into account the implications of his state of mind and choose the optimal one, obviously. "Nothing happens with a drink," "I will probably be regretful if I go to a bar, let me go home," "I will probably be regretful if I go to a bar, but let me go," "let me go and see what happens," these are all different behavioral outcomes implied by different states of mind.

There are different models in the fields of Economics that deal with these kind of situations, especially the bounded-rationality models. Regret models, addiction, procedural rationality and present bias with

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[†] Department of Economics, TOBB University of Economics and Technology, Sogutozu, Ankara, 06560, Turkey, +90 312 292 4216; zafer.akin@etu.edu.tr; http://www.etu.edu.tr/~zafer.akin

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sophisticated and naive beliefs are some examples. All of these models are motivated by limitations that the economic agents face (cognitive, ability-based etc.) and/or inherited behavioral characteristics of the agents.

In this paper, we focus on a very general approach that takes its underlying motivation from the *state of mind specification*. *State of mind* is a very general language and a comprehensive enough concept that covers any situation that one can think of (not only the economic decision problems, as the one above) and potentially embraces all the motivating figures of the models mentioned. By taking this specification as a benchmark, we propose a new equilibrium concept for a restricted class of one and two player extensive form games. As a first step, we take a simple and a deterministic approach that has its own dynamics and that can easily be extended to more general frameworks. A formalization of the motivating example and what do we mean by more general frameworks will be discussed shortly.

The rest of the paper is organized as follows. Section 2 discusses the equilibrium concept in detail for a restricted class of one player and two player games in order. Section 3 introduces a special case where the games played by potentially time-inconsistent agents are examined. Section 4 discusses some features of the existing model and an alternative approach as an extension. Section 5 concludes.

2 The Equilibrium Concept

2.1 Games with only one player

Let Γ be a generic extensive form game. There is only one player. A game tree is defined by the following objects: (N, A, Z) where $N : \{nodes\},\ A : \{actions\},\ Z : \{terminal\ nodes\}$. Let X be the set of decision nodes and $x_0 \in X$ be the initial node, $X = N \backslash Z,\ N = X \cup Z$. We assume perfect information (each information set is singleton). Let $a(x)$ be the set of actions from x for $x \in X$. Each node is reached by a single path. Let Γ^x be the subgame whose initial node is x, $x \in X$. Define N^x as the set of nodes including x and its successors. X^x and Z^x are defined by restrictions of X and Z to N^x. Define $A = \underset{x \in X}{\cup} a(x)$. Let $P(x)$ represent the set of predecessors of node x and $S(x) = X^x \backslash x$ is the set of all successors of node x.

Let Π be the set of agents' possible states of mind and it is inclusive enough that it captures all the different types of beliefs of the agents. Let $\pi_{x_0} \in \Pi$ be the initial state of mind (state of mind at node x_0) and it is given. Let u be the utility function and $u : \Pi \text{ x } Z \to \mathbb{R}$. Let Ψ be the conjectured transition function of the states of minds and $\Psi : \Pi \text{ x } A \to \Pi$. This means that the agent has potentially a different state of mind at each decision node $x \in X$ whether he moves at that node or not. Finally, let $\overline{\Psi}$ be the true transition function of the agents.

Some specifications:

a. Initial state of the agent, $\pi_{x_0} \in \Pi$, is given such that $u(\pi_{x_0}, z),\ \forall\ z \in Z$ is the utility function at x_0 and $\Psi(\pi_{x_0}, a),\ \forall\ a \in a(x_0)$ is the conjectured transition function.

b. At any node $\widehat{x}$, let $(x', x'') = a$ be an action in $a(x')$ where $x', x'' \in S(\widehat{x})$, then

The agent is in state $\pi_{\widehat{x}}$ and $\pi_{x''} = \Psi(\pi_{x'}, a)$ where $\pi_{x'}$ and $\pi_{x''}$ are the conjectured states at node x' and x'', respectively, from the perspective of node $\widehat{x}$ and they are determined by $\Psi(.,.)$ recursively. Moreover, the agent at $\widehat{x}$ thinks that $\pi_{x''}$ will be the state at node x'' from the perspective of all the nodes $x \in X^{\widehat{x}} \cap P(x'')$. In addition, at the node x', the actual state will be $\overline{\pi}_{x'}(i)$ that is determined by $\overline{\Psi}(.,.)$ recursively and $\pi_{x''} = \Psi(\overline{\pi}_{x'}, a)$.

c. $\overline{\pi}_{x''} = \overline{\Psi}(\overline{\pi}_{x'}, a)$

This specification means that at any node, the agent realizes his own true state and his true state in the next node is given by the true transition function, $\overline{\Psi}(.,.)$.

Current and future states are determined based on the above specifications. At each of his decision nodes x, given the state, the conjectured utility function and the transition function, the agent predicts how the game will proceed. Depending on this, he determines the conjectured path leading to the terminal node $(z^x)^*$. Then, at the node x, he chooses the action $a^* \in a(x)$ associated with this path.

The example:

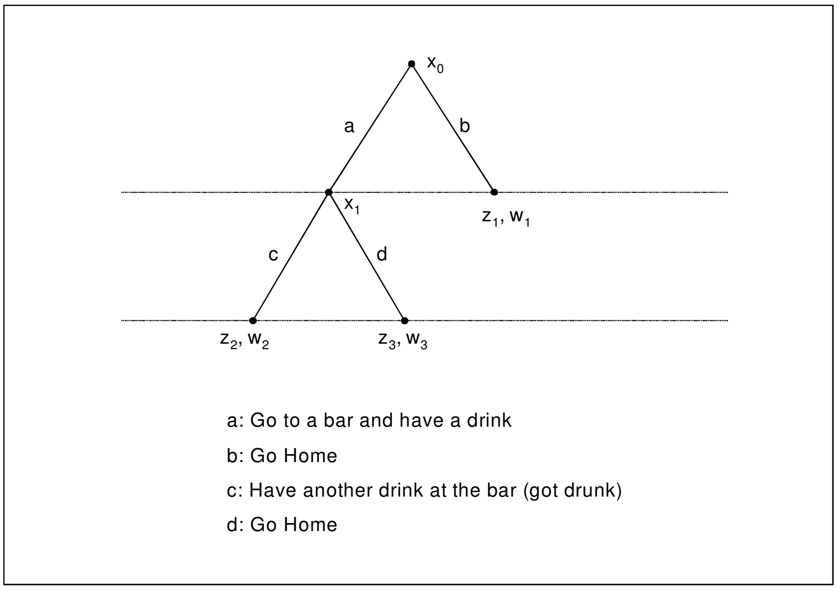

Figure 1:

In the above example of addiction, assume that there are three states, π_{01}, π_{02} and π_1. Conjectured transitions and actual transitions are given as follows:

$$\Psi(\pi_{01}, a) = \pi_{01};\ \Psi(\pi_{02}, a) = \pi_1;\ \overline{\Psi}(\pi_{01}, a) = \overline{\Psi}(\pi_{02}, a) = \pi_1$$

Utilities depending on the states satisfy the following:

$$\begin{aligned}
u(\pi_{01}, z_3) &> u(\pi_{01}, z_1) > u(\pi_{01}, z_2) \\
u(\pi_{02}, z_3) &> u(\pi_{02}, z_1) > u(\pi_{02}, z_2) \\
u(\pi_1, z_1) &> u(\pi_1, z_2) > u(\pi_1, z_3)
\end{aligned}$$

This example illustrates two addicts 1 and 2 having states π_{01} and π_{02}, respectively. Initially, both prefer having a drink before going home, z_3, to directly going home, z_1, that is preferred to get drunk, z_2. Agent 1 thinks that his preference ordering will not change if he goes to a bar, $\Psi(\pi_{01}, a) = \pi_{01}$. Agent 2 thinks that his preference ordering will change if he goes to a bar and he will keep drinking, $\Psi(\pi_{02}, a) = \pi_1$. So, for the second agent, (b, c) strategy is optimal that is going home directly. On the other hand, the first agent will go to a bar with the hope of having only one drink and then going home but he will end up getting drunk. It seems (a, d) strategy is optimal for him but he ends up following (a, c) which is the worse outcome according to his original state of mind[1].

Remark 1 *In the above specification, the states are not themselves the discounted utility or the beliefs about the future preferences. Instead, these are indicated by different states in the state (type) space* Π.

Remark 2 *At a given node, the agent has a given state. If it is the initial node, the state is exogenously given; if it is a decision node, the state is recursively assigned to that node by the conjectured transition function. This state implies what his utility function will be at that node. When he looks ahead and determines the future states from the perspective of any given node* x*, his perception on what his future states will be at each successor of* x *is based on the transition function and the state he has at node* x*.*

A Restricted Class of Perfect Information Extensive-Form Games

STEP 1: Games with 2 decision nodes

At x_0, agent believes that action a will result in his having state of mind $\Psi(\pi_{x_0}, a)$ at x_1 and action b will result in his having state of mind $\Psi(\pi_{x_0}, b)$ at z_1 and so that his preferences at x_1 and z_1 would be $u(\Psi(\pi_{x_0}, a), .)$ and $u(\Psi(\pi_{x_0}, b), .)$, respectively. However, when he reaches to these nodes, he will actually have preferences $u(\overline{\Psi}(\pi_{x_0}, a), .)$ and $u(\overline{\Psi}(\pi_{x_0}, b), .)$, respectively.

Assume that the agent does not gain any utility unless the game ends (there is no associated utility with any of the decision nodes). Let w_i be the outcome he gets if terminal node z_i is reached. Note that distinct nodes may have the same outcome. A special case is worth mentioning here. This is an agent where he has $(\delta, \beta, \widehat{\beta})$ where δ is time-consistent discount factor, β is time-inconsistent impatience and $\widehat{\beta}$ is his perception about the value of his β in the future. The intertemporal preferences for this special case can be represented by the following utility function:

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[1] Incorporating regret motives and probabilistic transitions would be a plausible extension.

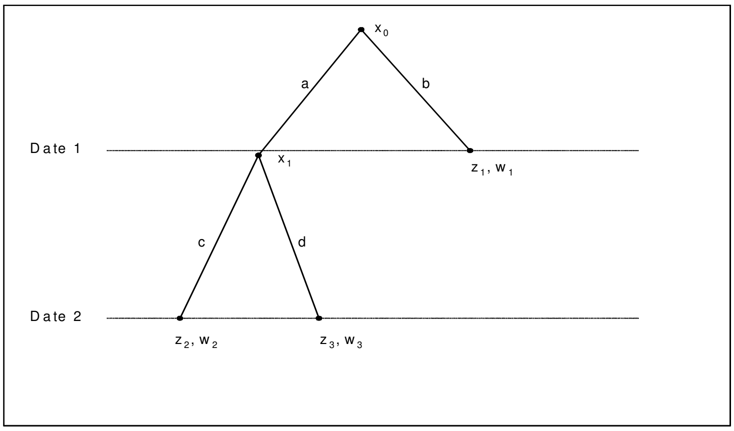

Figure 2:

At x_0, he has: $u(\pi_{x_0}, z_1) = \beta\delta w_1;\ u(\pi_{x_0}, z_2) = \beta\delta^2 w_2;\ u(\pi_{x_0}, z_3) = \beta\delta^2 w_3$

At x_0, he thinks at x_1 he will have: $u(\Psi(\pi_{x_0}, a), z_2) = \widehat{\beta}\delta w_2;\ u(\Psi(\pi_{x_0}, a), z_3) = \widehat{\beta}\delta w_3$

At x_1, he will actually have: $u(\overline{\Psi}(\pi_{x_0}, a), z_2) = \beta\delta w_2;\ u(\overline{\Psi}(\pi_{x_0}, a), z_3) = \beta\delta w_3$

This case will be examined later in detail. However, note that our definition allows more general cases.

Actual and conjectured transitions are shown in the following table:

	x_0	x_1
x_0	π_{x_0}	$\Psi(\pi_{x_0}, a)$
x_1	π_{x_0}	$\overline{\Psi}(\pi_{x_0}, a)$

The table shows the conjectured state of the agent at the column nodes from the perspective of the row nodes. For example, for the $x_0 - x_1$ entry in the table, the player is at node x_0 and he conjectures that his state at x_1 will be the corresponding entry, $\Psi(\pi_{x_0}, a)$[2]. Together with the above specifications, we can make the following definition of this game's equilibrium:

Definition 1 *A noncooperative equilibrium of the game above, defined by the set of states,* Π, *the set of actions,* A, *the utility function,* u, *the conjectured transition function,* Ψ, *and the true transition function of states,* $\overline{\Psi}$, *is a path from the initial node to a terminal node such that the initial state is* π_{x_0} *at the initial node* x_0 *and the terminal node is given by*

$$(z^{x_0})^* = \arg\max\{\ u(\pi_{x_0}, z_1), u(\pi_{x_0}, \arg\max\{u(\Psi(\pi_{x_0}, a), z_2),\ u(\Psi(\pi_{x_0}, a), z_3)\})\}$$

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[2] Note that lower diagonal elements are same with the diagonal element of the column that they belong to. In other words, $a_{ij} = a_{jj}\ \forall i > j$ where a_{ij} represents the element located at the row i and column j. This means that the agent remembers what his state was at all the previous nodes.

The actual path in the game does not have to be the same with the initially perceived one. The actual path will be the collection of actions such that

* Actual and conjectured state at node x_1 is determined based on the above specification.

* At each decision node $x \in X$, given the state and the conjectured transition function, the agent chooses the action $a^* \in a(x)$ associated with $(z^x)^*$ where

$$(z^{x_0})^* = \arg\max\{\ u(\pi_{x_0}, z_1), u(\pi_{x_0}, \arg\max\{u(\Psi(\pi_{x_0}, a), z_2),\ u(\Psi(\pi_{x_0}, a), z_3)\})\}$$

If $(z^{x_0})^* \neq z_1$ or x_1 is reached, then

$$(z^{x_1})^* = \arg\max\{u(\overline{\Psi}(\pi_{x_0}, a), z_2),\ u(\overline{\Psi}(\pi_{x_0}, a), z_3)\}$$

STEP 2: Games with three decision nodes

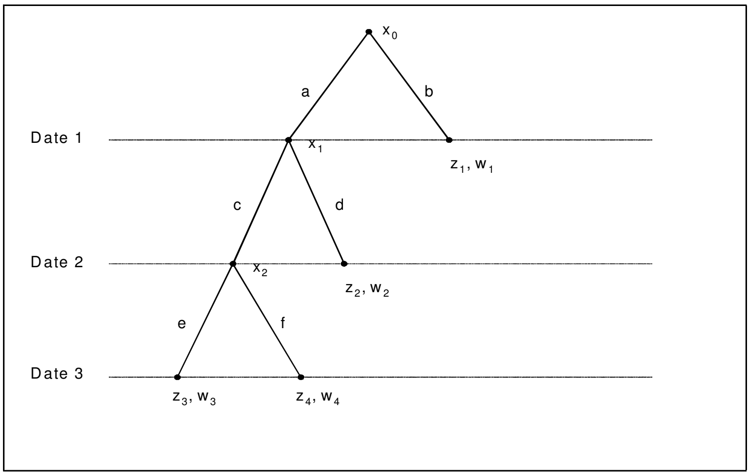

Figure 3:

At the example with three decision nodes, at x_0, agent believes that taking actions a and b will result in his having state of mind $\Psi(\pi_{x_0}, a)$ and $\Psi(\pi_{x_0}, b)$ at x_1 and z_1, so that his preferences at x_1 and z_1 would be $u(\Psi(\pi_{x_0}, a), .)$ and $u(\Psi(\pi_{x_0}, b), .)$, respectively. However, when he reaches to these nodes, he will actually have preferences $u(\overline{\Psi}(\pi_{x_0}, a), .)$ and $u(\overline{\Psi}(\pi_{x_0}, b), .)$, respectively. He also believes that taking actions c and d will result in his having state of mind $\Psi(\Psi(\pi_{x_0}, a), c)$ and $\Psi(\Psi(\pi_{x_0}, a), d)$ at x_2 and z_2, so that his preferences at x_2 and z_2 would be $u(\Psi(\Psi(\pi_{x_0}, a), c), .)$ and $u(\Psi(\Psi(\pi_{x_0}, a), d), .)$, respectively. However, when he reaches to these nodes, he will actually have preferences $u(\overline{\Psi}(\overline{\Psi}(\pi_{x_0}, a), c), .)$ and $u(\overline{\Psi}(\overline{\Psi}(\pi_{x_0}, a), c), .)$, respectively.

Again assume that the agent does not gain any utility unless the game ends (there is no associated utility with any of the decision nodes). Let w_i be the outcome he gets if terminal node z_i is reached. Actual and conjectured states are shown in the following table:

	x_0	x_1	x_2
x_0	π_{x_0}	$\Psi(\pi_{x_0}, a)$	$\Psi(\Psi(\pi_{x_0}, a), c)$
x_1	π_{x_0}	$\overline{\Psi}(\pi_{x_0}, a)$	$\Psi(\overline{\Psi}(\pi_{x_0}, a), c)$
x_2	π_{x_0}	$\overline{\Psi}(\pi_{x_0}, a)$	$\overline{\Psi}(\overline{\Psi}(\pi_{x_0}, a), c)$

Together with the above specifications, we can make the following definition of this game's equilibrium:

Definition 2 *A noncooperative equilibrium of the game above, defined by the set of states,* Π*, the set of actions,* A*, the utility function,* u*, the conjectured transition function,* Ψ*, and the true transition function of states,* $\overline{\Psi}$*, is a path from the initial node to a terminal node such that the initial state is* π_{x_0} *at the initial node* x_0 *and the terminal node is given by*

$$\begin{aligned} (z^{x_0})^* &= \arg\max\{\ u(\pi_{x_0}, z_1), u(\pi_{x_0}, \arg\max\{u(\Psi(\pi_{x_0}, a), z_2),\ u(\Psi(\pi_{x_0}, a), (z^{x_2})')\})\} \\ \textit{where } (z^{x_2})' &= \arg\max\{u(\Psi(\Psi(\pi_{x_0}, a), c), z_3),\ u(\Psi(\Psi(\pi_{x_0}, a), c), z_4)\} \end{aligned}$$

The actual path in the game does not have to be the same with the initially perceived one. The actual path will be the collection of actions such that

* Actual and conjectured state at node x_1 and x_2 is determined based on the above specification.

* At each decision node $x \in X$, given the state and the conjectured transition function, the agent chooses the action $a^* \in a(x)$ associated with $(z^x)^*$ where

$$\begin{aligned} (z^{x_0})^* &= \arg\max\{\ u(\pi_{x_0}, z_1), u(\pi_{x_0}, \arg\max\{u(\Psi(\pi_{x_0}, a), z_2),\ u(\Psi(\pi_{x_0}, a), (z^{x_2})')\})\} \\ \textit{where } (z^{x_2})' &= \arg\max\{u(\Psi(\Psi(\pi_{x_0}, a), c), z_3),\ u(\Psi(\Psi(\pi_{x_0}, a), c), z_4)\} \end{aligned}$$

If $(z^{x_0})^* \neq z_1$ or x_1 is reached, then

$$(z^{x_1})^* = \arg\max\{u(\overline{\Psi}(\pi_{x_0}, a), z_2),\ u(\overline{\Psi}(\pi_{x_0}, a), \arg\max\{u(\Psi(\overline{\Psi}(\pi_{x_0}, a), c), z_3),\ u(\Psi(\overline{\Psi}(\pi_{x_0}, a), c), z_4)\})\}$$

If $(z^{x_1})^* \neq z_2$ or x_2 is reached, then

$$(z^{x_2})^* = \arg\max\{u(\overline{\Psi}(\overline{\Psi}(\pi_{x_0}, a), c), z_3),\ u(\overline{\Psi}(\overline{\Psi}(\pi_{x_0}, a), c), z_4)\}$$

STEP T: Games with "T" decision nodes

Let Γ be a T period extensive form game as in the figure. There is only one player. To define a noncooperative equilibrium of Γ, we define two mappings m and g.

Function m, given any node, assigns a state to all successors of that node for all the nodes in the game, $m : X\text{x}N \to \Pi$ and is formally defined as

$$m(x, y) = \{ \begin{array}{c} \overline{\pi}_x,\ \text{if } y = x \\ \Psi(m(x, p(y)), a(p(y), y)),\ \text{if } y \in N^x \backslash \{x\} \end{array}$$

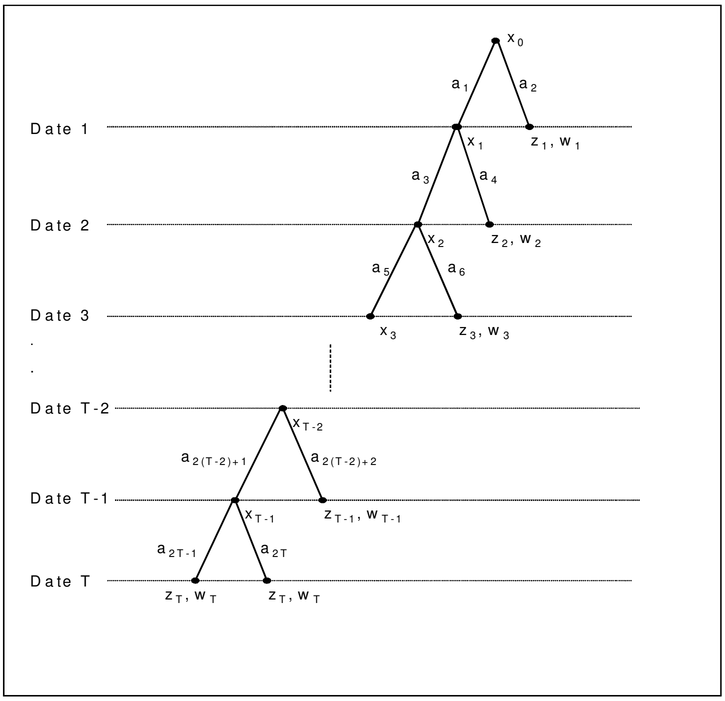

Figure 4:

where $a(x, y)$ is the action from node x that leads to node y; $p(y)$ is the immediate predecessor of y. In other words, $m(x, y)$ assigns a state to every node, $y \in N^x$, in the subgame starting with the node x from the perspective of node x. Note that if $y = x$, then the state that he assigns to the node where he is at is the true state, $\overline{\pi}_x$, of himself. Note also that if y is an immediate successor of x, then $p(y) = x$ and $m(x, y) = \Psi(m(x, p(y)), a(p(y), y)) = \Psi(m(x, x), a(x, y)) = \Psi(\overline{\pi}_x, a(x, y))$.

Remark 3 *The true state at any given node is exogenously given if it is the initial node, if not, it is determined by the true transition function* $\overline{\Psi}(.,.)$. *In the definition of* $m(x, y)$, *we did not specify what the agent's state is at node* x. *If the agent is actually at node* x *and assigning a state to a node* $y \in N^x$, *then his actual state at* x *is given by* $m(x_0, x) = \overline{\Psi}(\overline{\Psi}(....\overline{\Psi}(\overline{\Psi}(\pi_{x_0}, .), .)...., .), .)$. *This is different than the case where the agent is actually at some node* $x' \in P(x)$ *and assigning a state to node* x *(If* $x' = x_0$, $\pi_{x'} = \pi_{x_0}$ *is given; If* $x' \neq x_0$, $\pi_{x'}$ *is determined by* $\overline{\Psi}$ *recursively.) In that case, the conjectured state at* x *will be given by* $m(x', x) = \Psi(\Psi(....\Psi(\Psi(\pi_{x'}, .), .)...., .), .)$.

To see how function m works, we will give the following example. We want to find, at node x, what the conjectured state about node y_3 is (see the figure). This is given by $m(x, y_3)$. From the definition, we can write it as

$$m(x, y_3) = \Psi(m(x, y_2), a(y_2, y_3))$$

Now, we should find $m(x, y_2)$;

$$m(x, y_2) = \Psi(m(x, y_1), a(y_1, y_2))$$

To find $m(x, y_2)$, we need to find $m(x, y_1)$;

$$m(x, y_1) = \Psi(\pi_x, a(x, y_1))$$

Now, plug these recursively into $m(x, y_3)$ to get:

$$m(x, y_3) = \Psi(\Psi(\Psi(\pi_x, a(x, y_1)), a(y_1, y_2)), a(y_2, y_3))$$

Function $g : \Pi$ x $N \to \mathbb{Z}$ assigns a terminal node to all the nodes in the game given the state of the agent at that node. Suppose $\forall\ \pi \in \Pi,\ \forall y \in s(x),\ g(\pi, y)$ is defined where $s(x)$ is the set of immediate successors of node x. Then, g is defined as follows:

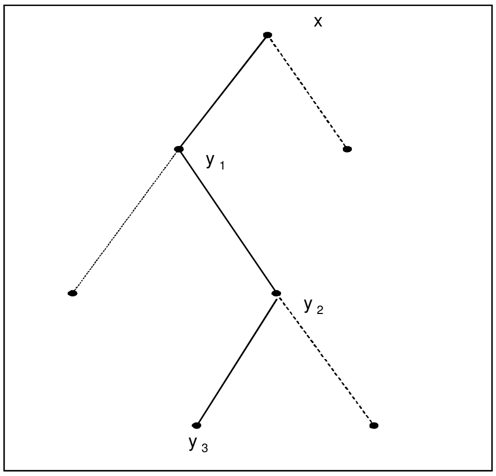

Figure 5:

$$g(\pi_x, x) = \{ \begin{array}{c} x,\ \text{if } x \in Z \\ \underset{y \in s(x)}{\arg\max}\{u(\pi_x, g(m(x, y), y))\},\ \text{if } x \in X \end{array}$$

Simply, function $g(.,.)$ assigns a terminal node to every decision node. If $\pi_x = \overline{\pi}_x$ ($\overline{\pi}_x :$ the actual state at node x implied by the true transition function, $\overline{\Psi}),$ then call $g(\pi_x, x)$ as $\overline{g}_x$.If $\pi_x = m(x', x)$ ($m(x', x) :$ the conjectured state at node x from the perspective of node $x' \in P(x)$, implied by the conjectured transition function, $\Psi),$ then call $g(\pi_x, x)$ as g_x. Let $a_{x_i}(x_j)$ be the action in $a(x_j)$ associated with the terminal node assigned to x_j by $g(\pi_{x_j}, x_j)$ from the perspective of node x_i where $\pi_{x_j} = m(x_i, x_j)$.

Based on the above definitions, a noncooperative equilibrium of a one player game can be defined as follows:

Definition 3 *A noncooperative equilibrium of a one player (finite period, extensive form) game, defined by the set of states,* Π*, the set of actions,* A*, the utility function,* u*, the transition function,* Ψ*, and the true transition function of states,* $\overline{\Psi}$*, is a path from the initial node to a terminal node such that the initial state is* π_{x_0} *at the initial node* x_0 *and the terminal node is* $\overline{g}_{x_0} = g(\pi_{x_0}, x_0)$ *defined above.*

Again the conjectured path from x_0 to $\overline{g}_x$ may not overlap with the actual path that will arise during the course of the game. This is true for any decision node $x \in X$. When the game proceeds, the agent's state of mind evolves based on the true transition function, so the actions that are associated with the terminal nodes $\overline{g}_{x_1}$, $\overline{g}_{x_2}$... from the nodes that are reached will be chosen. In other words, at any node x, the agent chooses the action $a_x(x) \in a(x)$ associated with $\overline{g}_x$ and $(x, a_x(x)) = y \in s(x)$ and $\overline{g}_x = g(\overline{\pi}_x, x) = g(\pi_y, y) = g(m(x,y), y) = z^* \in Z^x$ where y is an element of the set of decision nodes on the conjectured equilibrium path. However, at node y, it may be the case that $\overline{g}_y = g(\pi_y, y) = z^{**} \neq z^* \in Z^x$.

Remark 4 *Function* m *assigns a state to each node* x *by taking the transition function into account. The assignment of a state to node* x *by function* m *implicitly means that the utility function assigns a real number to the terminal nodes from the perspective of node* x*. Then, function* g *assigns a terminal node* z^* *to node* x*. The mappings are summarized in the following figure.*

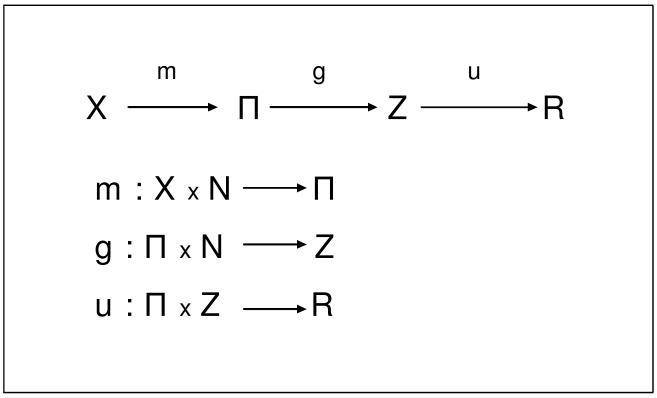

Figure 6:

To see how the function g recursively assigns every node to a terminal node, we will examine $T = 3$ case. The equilibrium for this game is the path from x_0 to $\overline{g}_{x_0} = g(\pi_{x_0}, x_0)$. We now find

$$\overline{g}_{x_0} = \arg\max_{y \in s(x_0)} \{u(\pi_{x_0}, g(m(x_0, y), y))\}$$

$$\overline{g}_{x_0} = \arg\max\{u(\pi_{x_0}, z_1), u(\pi_{x_0}, g(m(x_0, x_1), x_1))\} \tag{1}$$

To find the $\overline{g}_{x_0}$, we need to find first $g_{x_1} = g(m(x_0, x_1), x_1)$ as follows:

$$g_{x_1} = \underset{y \in s(x_1)}{\arg\max}\{u(m(x_0, x_1), g(m(x_0, y), y))\}$$

$$g_{x_1} = \arg\max\{u(\Psi(\pi_{x_0}, a_{x_1}(x_0)), z_2), u(\Psi(\pi_{x_0}, a_{x_1}(x_0)), g(m(x_0, x_2), x_2))\} \tag{2}$$

To find the g_{x_1}, we need to go one step further and find $g_{x_2} = g(m(x_0, x_2), x_2)$ as follows:

$$g_{x_2} = \underset{y \in s(x_2)}{\arg\max}\{u(\Psi(\Psi(\pi_{x_0}, a_{x_1}(x_0)), a_{x_2}(x_1)), g(m(x_0, y), y))\}$$

$$g_{x_2} = \arg\max\{u(\Psi(\Psi(\pi_{x_0}, a_{x_1}(x_0)), a_{x_2}(x_1)), z_3), u(\Psi(\Psi(\pi_{x_0}, a_{x_1}(x_0)), a_{x_2}(x_1)), z_4)\}$$

Then, plug g_{x_2} in (2) and get

$$g_{x_1} = \arg\max\{u(\Psi(\pi_{x_0}, a_{x_1}(x_0)), z_2),$$

$$u(\Psi(\pi_{x_0}, a_{x_1}(x_0)), \arg\max\{u(\Psi(\Psi(\pi_{x_0}, a_{x_1}(x_0)), a_{x_2}(x_1)), z_3), u(\Psi(\Psi(\pi_{x_0}, a_{x_1}(x_0)), a_{x_2}(x_1)), z_4)\}$$

Then, plug g_{x_1} in (1) and get $g_{x_0} = g(\pi_{x_0}, x_0)$

$$g_{x_0} = g(\pi_{x_0}, x_0) = \arg\max\{u(\pi_{x_0}, z_1), u(\pi_{x_0}, \arg\max\{u(\Psi(\pi_{x_0}, a_{x_1}(x_0)), z_2),$$

$$u(\Psi(\pi_{x_0}, a_{x_1}(x_0)), \arg\max\{u(\Psi(\Psi(\pi_{x_0}, a_{x_1}(x_0)), a_{x_2}(x_1)), z_3), u(\Psi(\Psi(\pi_{x_0}, a_{x_1}(x_0)), a_{x_2}(x_1)), z_4)\}$$

Thus, the mapping g allows us to extend the definition of the equilibrium to the games with T decision nodes.

2.2 Games with two players

Let Γ be a generic extensive form game. There are two players, $i \neq j \in \{1, 2\}$. A game tree is defined by the following objects: (N, A, Z) where $N : \{nodes\}$, $A : \{actions\}$, $Z : \{terminal\ nodes\}$. Let X be the set of decision nodes, $x_0 \in X$ be the initial node and $X = N \backslash Z$, $N = X \cup Z$. We assume perfect information (each information set is singleton). The function λ assigns each nonterminal node to a player, $\lambda : X \to \{i, j\}$. Let $a(x)$ be the set of actions from x for $x \in X$. In other words, $a(x)$ is the set of moves for $\lambda(x)$ at x. Each node is reached by a single path. Let Γ^x be the subgame whose initial node is x for $x \in X$. Define N^x as the set of nodes including x and its successors. X^x and Z^x are defined by restrictions of X and Z to N^x. Define $A = \underset{x \in X}{\cup} a(x)$. Let $P(x)$ represent the set of all predecessors of node x and $S(x) = X^x \backslash x$ is the set of all successors of node x.

Let Π be the set of agents' possible states of mind and it is inclusive enough that it captures all the different types of beliefs of the agents. Let $\pi_{x_0}(i) \in \Pi$ be the initial state of mind of agent i and it is privately known by agent i. Let u be the utility function and $u : \Pi \text{ x } Z \to \mathbb{R}$. Let Ψ be the conjectured transition function of the states of minds and $\Psi : \Pi \text{ x } A \rightarrow \Pi$. This means that each agent has potentially a different state of mind at each decision node $x \in X$ whether he moves at that node or not. Let l be a function that maps state space to itself, $l : \Pi \rightarrow \Pi$. In words, $l(\pi(i)) = \pi(j)$ is $i's$ conjecture about what $j's$ state is. Finally, let $\overline{\Psi}$ be the true transition function of the agents. Function u, l and Ψ are common knowledge but $\overline{\Psi}$ is not.

Some specifications:

a. Initial state of the agent i, $\pi_{x_0}(i) \in \Pi$, is given such that $u(\pi_{x_0}(i), z)$, $\forall\ z \in Z$ is the utility function of agent i at x_0; $\Psi(\pi_{x_0}(i), a)$, $\forall\ a \in a(x_0)$ is the conjectured transition function of agent i and $l(\pi_{x_0}(i))$ is the function that assigns a state to agent j given $i's$ initial state $\pi_{x_0}(i)$.

b. $l(\pi_{x_0}(i)) = \pi^i_{x_0}(j)$: $i's$ conjecture about the initial state of j,

$u(\pi^i_{x_0}(j), .)$: $i's$ conjecture about the valuation of terminal nodes of agent j from the perspective of node x_0,

$\Psi(\pi^i_{x_0}(j), .)$: $i's$ conjecture about agent $j's$ states at the immediate successors of x_0 from the perspective of node x_0.

c. At any node $\widehat{x}$, let $(x', x'') = a$ be an action in $a(x')$ where $x', x'' \in X^{\widehat{x}}$, then

Agent i is in state $\pi_{\widehat{x}}(i)$. Then, $\pi_{x''}(i) = \Psi(\pi_{x'}(i), a)$ where $\pi_{x'}(i)$ and $\pi_{x''}(i)$ are the conjectured states at node x' and x'' agent i, respectively, from the perspective of node $\widehat{x}$ and they are determined by $\Psi(.,.)$ recursively. Moreover, the agent i at $\widehat{x}$ thinks that $\pi_{x''}(i)$ will be his state at node x'' from the perspective of all the nodes $x \in X^{\widehat{x}} \cap P(x'')$. In addition, at the node x', the actual state of the agent i will be $\overline{\pi}_{x'}(i)$ that is determined by $\overline{\Psi}(.,.)$ recursively and $\pi_{x''}(i) = \Psi(\overline{\pi}_{x'}(i), a)$.

d. At any node $\widehat{x}$, let $(x', x'') = a$ be an action in $a(x')$ where $x', x'' \in X^{\widehat{x}}$, then

Agent i is in state $\pi_{\widehat{x}}(i)$. This implies $l(\pi_{\widehat{x}}(i)) = \pi^i_{\widehat{x}}(j)$. Furthermore, $\pi^i_{x''}(j) = \Psi(\pi^i_{x'}(j), a)$ where $\pi^i_{x'}(j)$ is i's conjecture about the state of agent j at node x' from the perspective of node $\widehat{x}$ and it is determined by $\Psi(.,.)$ recursively and $\pi^i_{x''}(j)$ is the node $\widehat{x}$ conjecture of the agent i about node x'' state of agent j. Moreover, agent i at $\widehat{x}$ thinks that $\pi^i_{x''}(j)$ will be the j's state at node x'' from the perspective of all the nodes $x \in X^{\widehat{x}} \cap P(x'')$. In addition, at the node x', the actual state of the agent i will be $\overline{\pi}_{x'}(i)$ that is determined by $\overline{\Psi}(.,.)$ recursively and then $l(\overline{\pi}_{x'}(i)) = \overline{\pi}^i_{x'}(j)$ and $\overline{\pi}^i_{x''}(j) = \Psi(\overline{\pi}^i_{x'}(j), a)$. Moreover, both $\overline{\pi}^i_{x'}(j)$ and $\overline{\pi}^i_{x''}(j)$ are potentially different than the ones conjectured at node $\widehat{x}$.

e. $\overline{\pi}_{x''}(i) = \overline{\Psi}(\overline{\pi}_{x'}(i), a)$

This specification means that at any node, each agent realizes his own true state and his true state in the next node is given by the true transition function, $\overline{\Psi}(.,.)$.

True and conjectured states are determined based on the above specifications. Each agent, at each of the decision node x, given his state, his conjectured utility function, transition function and the conjectured state of the opponent, predicts how the game will proceed. Depending on this, he determines the conjectured path leading to the terminal node $(z^x)^*$. Then, at the node x, he chooses the action $a^* \in a(x)$ associated with this path.

Remark 5 *In the above specification, the states are not themselves the discounted utility or the beliefs about the future preferences or the beliefs about the opponent's state. Instead, these are implied by different states in the state (type) space* Π.

Remark 6 *When the agent i looks ahead from a node x and determines his own future states, his perception on what his future states will be at each successor of x is based on the transition function and the state he has at node x,* $\Psi(\pi_x(i), .)$. *His perception on what his opponent's state will be at each successor of x is based on the transition function and the state that he assigns to his opponent at node x,* $l(\pi_x(i)) = \pi_x^i(j)$ *and* $\Psi(\pi_x^i(j), .)$.

Without loss of generality, in every case, we will assume that player 1 is the first mover.

A Restricted Class of Perfect Information Extensive-Form Games

STEP 1: Each player has only one decision node

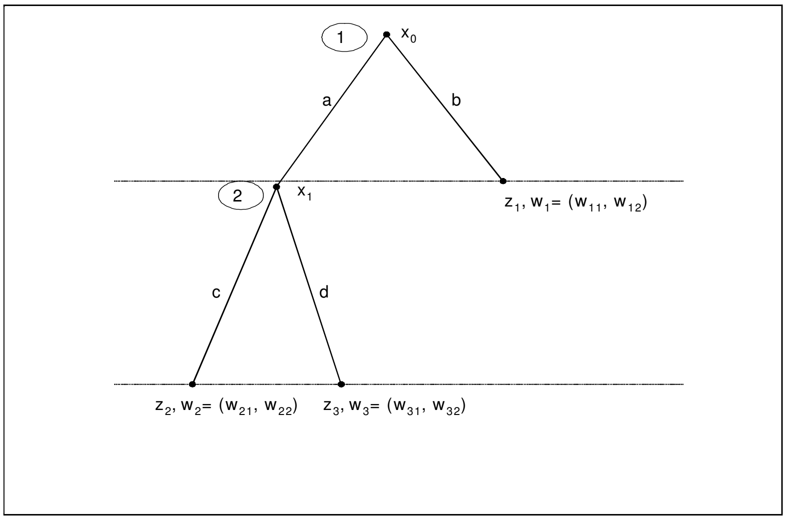

Figure 7:

Initial states of agent 1 and 2 are $\pi_{x_0}(1)$ and $\pi_{x_0}(2)$, respectively. This implies that taking action a (b) will cause agent 1 to have state of mind $\Psi(\pi_{x_0}(1), a)$ at x_1 ($\Psi(\pi_{x_0}(1), b)$ at z_1). The initial state of 1, $\pi_{x_0}(1)$, also implies that according to agent 1, agent 2 has state $l(\pi_{x_0}(1)) = \pi_{x_0}^1(2)$ and has utility function $u(\pi_{x_0}^1(2), .)$ at node x_0.

Assume that the agent does not gain any utility unless the game ends (there is no associated utility with any of the decision nodes). Let $w_k = (w_{k1}, w_{k2})$, where w_{ki} is the outcome of agent i, be the outcome vector if terminal node z_k is reached. Note that distinct nodes may have the same outcome. Together with the above specifications, we can make the following definition of this game's equilibrium:

Definition 4 *A noncooperative equilibrium of the game with each player having one decision node, defined by the set of states,* Π*, the set of actions,* A*, state dependent utility function,* $u(.,.)$*, the transition function,* $\Psi(.,.)$*, the function* $l(.)$ *and the true transition function,* $\overline{\Psi}(.,.)$*, is a path from the initial node to a terminal node such that the initial state is* $\pi_{x_0}(i)$ *at the initial node* x_0 *and the terminal node is given by*

$$(z^{x_0})^* = \arg\max\{\ u(\pi_{x_0}(1), z_1), u(\pi_{x_0}(1), \arg\max\{u(\Psi(l(\pi_{x_0}(1)), a), z_2),\ u(\Psi(l(\pi_{x_0}(1)), a), z_3)\}\})\}$$

The actual path in the game does not have to be the same with the initially perceived one. The actual path will be the collection of actions such that

* Actual and conjectured state at node x_1 is determined based on the above specification.

* At each decision node $x \in X$, agents choose the action $a^* \in a(x)$ associated with $(z^x)^*$ where

$$(z^{x_0})^* = \arg\max\{\ u(\pi_{x_0}(1), z_1), u(\pi_{x_0}(1), \arg\max\{u(\Psi(l(\pi_{x_0}(1)), a), z_2),\ u(\Psi(l(\pi_{x_0}(1)), a), z_3)\}\})\}$$

If $(z^{x_0})^* \neq z_1$ or x_1 is reached, then

$$(z^{x_1})^* = \arg\max\{u(\overline{\Psi}(\pi_{x_0}(2), a), z_2),\ u(\overline{\Psi}(\pi_{x_0}(2), a), z_3)\}$$

STEP 2: One player has one decision node, the other has two decision nodes

Initial states of agent 1 and 2 are $\pi_{x_0}(1)$ and $\pi_{x_0}(2)$, respectively. This implies that taking action a (b) will cause agent 1 to have state of mind $\Psi(\pi_{x_0}(1), a)$ at x_1 ($\Psi(\pi_{x_0}(1), b)$ at z_1). The initial state of 1, $\pi_{x_0}(1)$, also implies that according to agent 1, agent 2 has state $l_1(\pi_{x_0}(1)) = \pi^1_{x_0}(2)$ and has utility function $u(\pi^1_{x_0}(2), .)$ at node x_0.

Agent 1 will make backwards induction as follows: if node x_2 is reached, he will be in state $\Psi(\Psi(\pi_{x_0}(1), a), c)$ and $u(\Psi(\Psi(\pi_{x_0}(1), a), c), z_3)$ and $u(\Psi(\Psi(\pi_{x_0}(1), a), c), z_4)$ are the associated utilities with the terminal nodes z_3 and z_4, respectively. However, this assessment of agent 1 is relevant only if agent 1 believes that agent 2 will choose action c at node x_1. To see this, agent 1 has to conjecture about what agent 2 thinks what agent 1 will do at node x_2. Agent 1 believes that agent 2 will be in state $\Psi((l(\pi_{x_0}(1)), a)$ at node x_1. Then agent 1 believes that agent 2 believes that agent 1 will be at state $\Psi(l(\Psi((l(\pi_{x_0}(1)), a)), c)$ at node x_2. So, agent 1 believes that agent 2 believes that agent 1 will choose z_3 if

$$u(\Psi(l(\Psi((l(\pi_{x_0}(1)), a)), c), z_3) \geq u(\Psi(l(\Psi((l(\pi_{x_0}(1)), a)), c), z_4)$$

at node x_2 and vice versa. Then, agent 1 believes that agent 2 will compare the chosen terminal node and z_2 given agent $2's$ conjectured state according to agent 1, which is $\Psi(l(\pi_{x_0}(1)), a)$. If agent 1 believes that agent 2 will choose z_2, then agent 1 compares z_1 and z_2 given his initial state $\pi_{x_0}(1)$. If agent 1 believes that

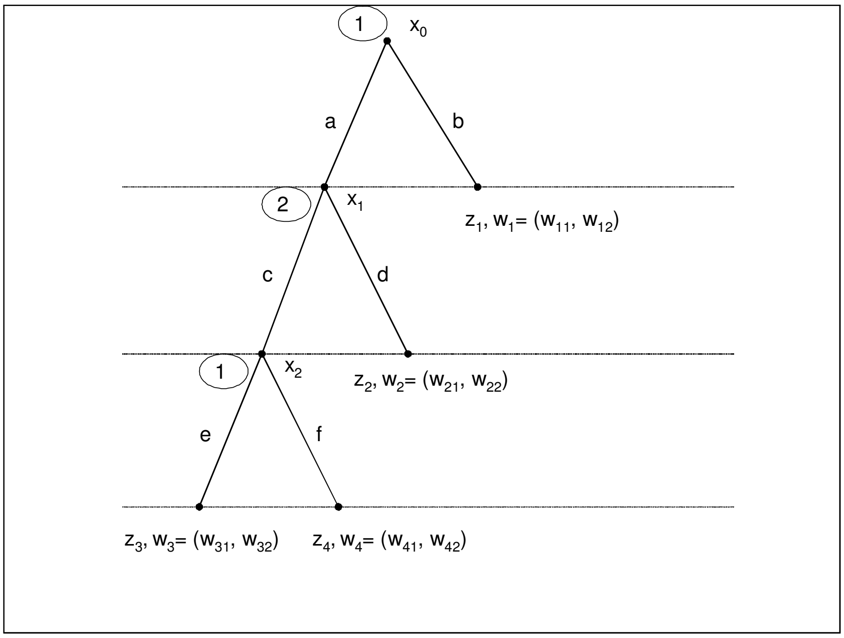

Figure 8:

agent 2 will not choose z_2, then agent 1 compares z_1 and what he thinks he will actually choose at x_2 (z_3 or z_4) given his initial state $\pi_{x_0}(1)$. What he thinks he will actually choose at x_2 is z_3 if

$$u(\Psi(\Psi(\pi_{x_0}(1), a), c), z_3) \geq u(\Psi(\Psi(\pi_{x_0}(1), a), c), z_4)$$

and vice versa.

We can formalize what is mentioned in the last paragraph as follows and this actually gives the definition of the equilibrium:

Definition 5 *A noncooperative equilibrium of the game with one player having one and the other having two decision nodes, defined by the set of states,* Π, *the set of actions,* A, *state dependent utility function,* $u(.,.)$, *the transition function,* $\Psi(.,.)$, *the function* $l(.)$ *and the true transition function,* $\overline{\Psi}(.,.)$, *is a path from the initial node to a terminal node such that the initial states are* $\pi_{x_0}(i)$ *at the initial node* x_0 *and the terminal node is given by*

$$(z^{x_0})^* = \arg\max\{\ u(\pi_{x_0}(1), z_1),\ \ u(\pi_{x_0}(1), \widehat{z})\}$$

$$\begin{aligned} z^{x_1} \quad &= \quad \arg\max\{u(\Psi(l(\pi_{x_0}(1)), a), z_2), \\ &\qquad u(\Psi(l(\pi_{x_0}(1)), a),\ \ \arg\max\{u(\Psi(l(\Psi((l(\pi_{x_0}(1)), a)), c), z_3),\ \ u(\Psi(l(\Psi((l(\pi_{x_0}(1)), a)), c), z_4)\})\} \end{aligned}$$

$$z^{x_2} = \arg\max\{u(\Psi(\Psi(\pi_{x_0}(1), a), c), z_3),\ \ u(\Psi(\Psi(\pi_{x_0}(1), a), c), z_4)\}$$

$$\begin{aligned} If\ z^{x_1} &= z_2,\ then\ \widehat{z} = z^{x_1} \\ If\ z^{x_1} &\neq z_2,\ then\ \widehat{z} = z^{x_2} \end{aligned}$$

The actual path in the game does not have to be the same with the initially perceived one. The actual path will be the collection of actions such that

* Actual and conjectured states at each node are determined based on the above specifications.

* At each decision node $x \in X$, agents choose the action $a^* \in a(x)$ associated with $(z^x)^*$ where $(z^{x_0})^*$ is defined above; if $(z^{x_0})^* \neq z_1$ or x_1 is reached, then the action $a^* \in a(x_1)$ associated with $(z^{x_1})^*$ where $(z^{x_1})^*$ is defined as

$$\begin{aligned} (z^{x_1})^* &= \arg\max\{u(\overline{\Psi}(\pi_{x_0}(2), a), z_2), \\ &\quad u(\overline{\Psi}_2(\pi_{x_0}(2), a),\ \arg\max\{u(\Psi(l(\overline{\Psi}(\pi_{x_0}(2), a)), c), z_3),\ u(\Psi(l(\overline{\Psi}(\pi_{x_0}(2), a)), c), z_4)\})\} \end{aligned}$$

If $(z^{x_1})^* \neq z_2$ or x_2 is reached, then the action $a^* \in a(x_2)$ associated with $(z^{x_2})^*$ where $(z^{x_2})^*$ is defined as

$$(z^{x_2})^* = \arg\max\{u(\overline{\Psi}(\overline{\Psi}(\pi_{x_0}(1), a), c), z_3),\ u(\overline{\Psi}(\overline{\Psi}(\pi_{x_0}(1), a), c), z_4)\}$$

STEP 3: Each player has two decision nodes

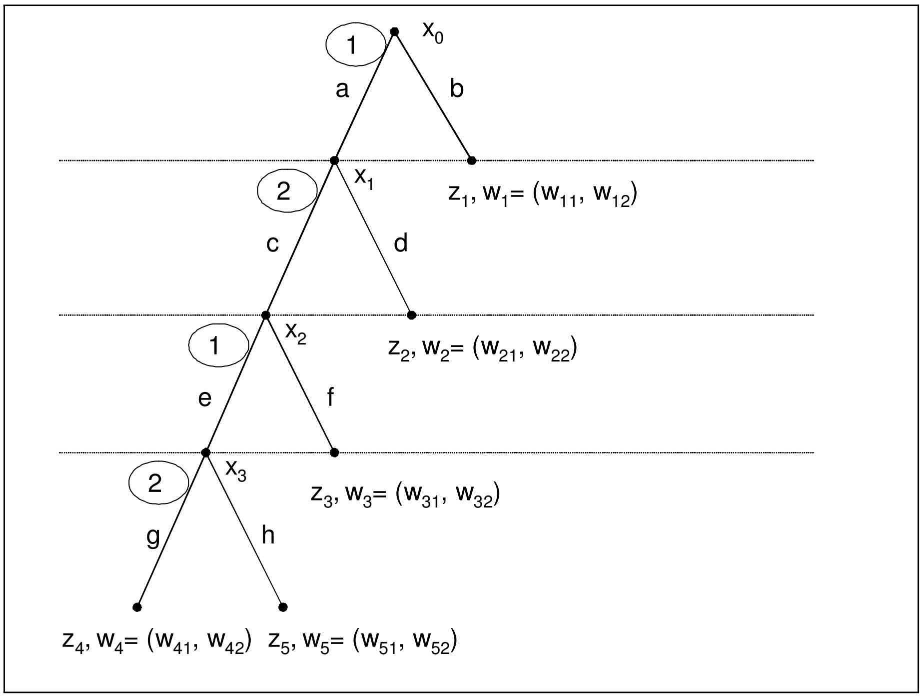

Figure 9:

This step is a little more complicated because there are different cases that may arise. To define the equilibrium, we have to analyze what each agent believes about how the game will evolve by keeping in mind that the game may actually evolve differently. In the previous step (step 2), player 1 looked ahead and put himself in the place of player 2 and conjectured about what player 2 will choose. If player 1 conjectures that he will choose d and end the game, then player 1 compares terminal nodes z_1 and z_2. If player 1 conjectures that player 2 will choose c, then player 1 compares z_1 and the terminal node that he thinks he will actually choose at node x_2 (z_3 or z_4). This is a similar argument to forward induction.

In this step, the arguments are more complicated because there more cases to analyze. For each case in step 2, there are two corresponding cases in this step that makes total four cases. There are two cases for both of the followings, if player 1 thinks that player 2 thinks that player 1 will choose f and not choose f at node x_2. For the first case, player 1 thinks that the optimal thing to do for player 2 is to choose c at node x_1 by expecting player 1 to choose f after that or to choose d at node x_1 (these two refer to case 1 and 2 below, respectively.) For the second case, player 1 thinks that the optimal thing to do for player 2 is to choose d at node x_1 or choose c at node x_1 by expecting player 1 not to play f at node x_2 (these two refer to case 3 and 4 below, respectively.)

In the light of the above argument, we can write the formal definition of the equilibrium for the above game as follows:

Definition 6 *A noncooperative equilibrium of the game with both players having two decision nodes, defined by the set of states,* Π, *the set of actions,* A, *state dependent utility function,* $u(.,.)$, *the transition function,* $\Psi(.,.)$, *the function* $l(.)$ *and the true transition function,* $\overline{\Psi}(.,.)$, *is a path from the initial node to a terminal node such that the initial states are* $\pi_{x_0}(i)$ *at the initial node* x_0 *and the terminal node is given by*

$$(z^{x_0})^* = \arg\max\{\ u(\pi_{x_0}(1), z_1),\ u(\pi_{x_0}(1), \widehat{z})\}$$

$$z^{x_1} = \arg\max\{u(\Psi(l(\pi_{x_0}(1)), a), z_2),\ u(\Psi(l(\pi_{x_0}(1)), a),\ z^{x_2})\}$$

$$\begin{aligned} z^{x_2} \quad = \quad & \arg\max\{u(\Psi(l(\Psi(l(\pi_{x_0}(1)), a)), c), z_3),\ u(\Psi(l(\Psi(l(\pi_{x_0}(1)), a)), c), \\ & \arg\max\{u(\Psi(l(\Psi(l(\Psi(l(\pi_{x_0}(1)), a)), c)), e), z_4),\ u(\Psi(l(\Psi(l(\Psi(l(\pi_{x_0}(1)), a)), c)), e), z_5)\})\} \end{aligned}$$

$$\begin{aligned} z^{x'_1} \quad = \quad & \arg\max\{u(\Psi(l(\pi_{x_0}(1)), a), z_2), \\ & u(\Psi(l(\pi_{x_0}(1)), a),\ \ \arg\max\{u(\Psi(\Psi(\Psi(l(\pi_{x_0}(1)), a), c), e), z_4),\ u(\Psi(\Psi(\Psi(l(\pi_{x_0}(1)), a), c), e), z_5)\})\} \end{aligned}$$

$$\begin{aligned} z^{x'_2} \quad = \quad & \arg\max\{u(\Psi(\Psi(\pi_{x_0}(1), a), c), z_3), \\ & u(\Psi(\Psi(\pi_{x_0}(1), a), c),\ \ \arg\max\{u(\Psi(l(\Psi(\Psi(\pi_{x_0}(1), a), c)), e), z_4),\ u(\Psi(l(\Psi(\Psi(\pi_{x_0}(1), a), c)), e), z_5)\})\} \end{aligned}$$

$$\begin{aligned}
1.\ If\ z^{x_2} &= z_3\ and\ z^{x_1} = z_3,\ then\ \widehat{z} = z^{x_2'} \\
2.\ If\ z^{x_2} &= z_3\ and\ z^{x_1} = z_2,\ then\ \widehat{z} = z^{x_1} \\
3.\ If\ z^{x_2} &\neq z_3\ and\ z^{x_1'} = z_2,\ then\ \widehat{z} = z^{x_1'} \\
4.\ If\ z^{x_2} &\neq z_3\ and\ z^{x_1'} \neq z_2,\ then\ \widehat{z} = z^{x_2'}
\end{aligned}$$

Again the actual path is not essentially the same with the above one. If $(z^{x_0})^* = z_1$, then the game ends immediately. However, if $(z^{x_0})^* \neq z_1$, then the actual path can be found easily by changing the initial state in step 2 accordingly.

The generalization of the equilibrium concept for two player games as we did in one player case needs a very careful examination at different levels of the game tree and remains to be done. Not having a general form of the equilibrium concept for two player games does not prevent us to generalize it in the frameworks where the states imply simple state transitions and trivial assignments of states to opponents. This is actually the case in so called *"Naive Backwards Induction"* solution concept that will be explained in the next section and it is a special case of the equilibrium concept presented here. Since the paper by Akin (2005) uses this solution concept, the analysis in the next section should be enough for our purposes.

3 A Special Case of Time-Inconsistent Preferences

In this part, we will illustrate how the quasi-hyperbolic model (Laibson, 1997; Phelps-Pollak, 1968; Strotz, 1956) fits in the constructed framework. The preference structure is as follows: there are possibly four types of players: exponential (EA), Naive hyperbolic (NHA), Sophisticated hyperbolic (SHA) and partially naive hyperbolic (PNHA). The EA has the following sequence of discount factors: $\{1, \delta, \delta^2, \delta^3, ...\}$. The NHA, the SHA and the PNHA all have the following same sequence of discount factors: $\{1, \beta\delta, \beta\delta^2, \beta\delta^3, ...\}$ where δ is the standard time-consistent impatience with $\delta \in (0, 1)$, β is time-inconsistent preference for immediate gratification or the self-control problem of the agent with $\beta \in (0, 1)$. $\widehat{\beta}$ represents a person's belief about his future self-control problems. In other words, $\widehat{\beta}$ is his belief about what his β will be in all future periods. The NHA believes he will not have self-control problems in the future, therefore, $\widehat{\beta} = 1$. The SHA knows exactly what future self-control problems will be, therefore, $\widehat{\beta} = \beta$. The partially naive person has perceptions $\widehat{\beta} \in (\beta, 1)$.

In the state of mind specification, there are more types than the ones specified above. The above specification does not take second order beliefs as part of the type. However, in our specification, there may be different naive types, different exponential types and so on. For our purposes, we introduce first the exponential, naive and sophisticated states as follows:

$$\begin{aligned}
\pi(N) \quad &\Rightarrow \quad u(\pi(N), w^t) = w_0 + \beta \sum_{i=1}^{t} \delta^i w_i \text{ where } w^t = \{w_j\}_{j=0}^{j=t} \\
&\Rightarrow \quad \Psi(\pi(N), a) = \overline{\Psi}(\pi(N), a) = \pi(N) \ \forall \ a \in A \text{ that occurs in the same period} \\
&\Rightarrow \quad \Psi(\pi(N), a) = \pi(E_2); \ \overline{\Psi}(\pi(N), a) = \pi(N) \ \forall \ a \in A \text{ that leads to the next period} \\
&\Rightarrow \quad l(\pi(N)) = \pi(E_1)
\end{aligned}$$

$$\begin{aligned}
\pi(E_1) \quad &\Rightarrow \quad u(\pi(E_1), w^t) = w_0 + \sum_{i=1}^{t} \delta^i w_i \text{ where } w^t = \{w_j\}_{j=0}^{j=t} \\
&\Rightarrow \quad \Psi(\pi(E_1), a) = \overline{\Psi}(\pi(E_1), a) = \pi(E_2) \ \forall \ a \in A \\
&\Rightarrow \quad l(\pi(E_1)) = \pi(N)
\end{aligned}$$

$$\begin{aligned}
\pi(E_2) \quad &\Rightarrow \quad u(\pi(E_2), w^t) = w_0 + \sum_{i=1}^{t} \delta^i w_i \text{ where } w^t = \{w_j\}_{j=0}^{j=t} \\
&\Rightarrow \quad \Psi(\pi(E_2), a) = \overline{\Psi}(\pi(E_2), a) = \pi(E_2) \ \forall \ a \in A \\
&\Rightarrow \quad l(\pi(E_2)) = \pi(E_2)
\end{aligned}$$

$$\begin{aligned}
\pi(\overline{E}) \quad &\Rightarrow \quad u(\pi(\overline{E}), w^t) = w_0 + \sum_{i=1}^{t} \delta^i w_i \text{ where } w^t = \{w_j\}_{j=0}^{j=t} \\
&\Rightarrow \quad \Psi(\pi(\overline{E}), a) = \overline{\Psi}(\pi(\overline{E}), a) = \pi(\overline{E}) \ \forall \ a \in A \\
&\Rightarrow \quad l(\pi(\overline{E})) = \pi(N)
\end{aligned}$$

$$\begin{aligned}
\pi(S) \quad &\Rightarrow \quad u(\pi(S), w^t) = w_0 + \beta \sum_{i=1}^{t} \delta^i w_i \text{ where } w^t = \{w_j\}_{j=0}^{j=t} \\
&\Rightarrow \quad \Psi(\pi(S), a) = \overline{\Psi}(\pi(S), a) = \pi(S) \ \forall \ a \in A \\
&\Rightarrow \quad l(\pi(S)) = \pi(\widehat{E})
\end{aligned}$$

$$\begin{aligned}
\pi(\widehat{E}) \quad &\Rightarrow \quad u(\pi(\widehat{E}), w^t) = w_0 + \sum_{i=1}^{t} \delta^i w_i \text{ where } w^t = \{w_j\}_{j=0}^{j=t} \\
&\Rightarrow \quad \Psi(\pi(\widehat{E}), a) = \overline{\Psi}(\pi(\widehat{E}), a) = \pi(\widehat{E}) \ \forall \ a \in A \\
&\Rightarrow \quad l(\pi(\widehat{E})) = \pi(S)
\end{aligned}$$

The agent with initial state $\pi(N)$ is naive as defined at the beginning of the section. He believes that for any action taken during the game, he will move to a kind of an exponential state, $\pi(E_2)$ and he believes that his opponent is another kind of an exponential agent with state $\pi(E_1)$. The exponential agent whose type is $\pi(E_1)$ is exponential in his discounting. He believes that he will move to another kind of an exponential state, $\pi(E_2)$ for any action and he believes that his opponent's state is $\pi(N)$. The exponential agent whose type is $\pi(E_2)$ is exponential in his discounting. He believes that he will stay in the same state, $\pi(E_2)$ for any action and he believes that his opponent's state is same with himself. The exponential agent whose type is $\pi(\overline{E})$ is the "actual" exponential agent who believes that he will stay in the same state, $\pi(\overline{E})$ for any action and believes that his opponent's state is $\pi(N)$.

On the other hand, the agent with initial state $\pi(S)$ is sophisticated as defined at the beginning of the section. He believes that for any action taken during the game, he will stay in the same state $\pi(S)$ and he also believes that his opponent is an exponential agent with state $\pi(\widehat{E})$. The exponential agent whose type is $\pi(\widehat{E})$ believes that he will stay in the same state, $\pi(\widehat{E})$ for any action and he believes that his opponent is sophisticated, $\pi(S)$. Thus, the states are common knowledge in the game between these two types.

Now, we introduce the partially naive state as follows:

$$\begin{aligned}
& & & \pi(PN_{m,n}) \\
& & \Rightarrow\ & u(\pi(PN_{m,n}), w^t) = w_0 + \beta\sum_{i=1}^{t}\delta^i w_i \text{ where } w^t = \{w_j\}_{j=0}^{j=t} \\
& & \Rightarrow\ & \Psi(\pi(PN_{m,n}), a) = \overline{\Psi}(\pi(PN_{m,n}), a) = \pi(PN_{m,n}) \\
\forall\ a & \in & & A \text{ that occurs in the same period} \\
& & \Rightarrow\ & \Psi(\pi(PN_{m,n}), a) = \pi(PE_{m,n});\ \overline{\Psi}(\pi(PN_{m,n}), a) = \pi(PN_{m,n}) \\
\forall\ a & \in & & A \text{ that leads to the next period} \\
& & \Rightarrow\ & l(\pi(PN_{m,n})) = \pi(PE_1)
\end{aligned}$$

$$\begin{aligned}
& & & \pi(PE_{m,n}) \\
& & \Rightarrow\ & u(\pi(PE_{m,n}), w^t) = w_0 + \widehat{\beta}_{m,n}\sum_{i=1}^{t}\delta^i w_i \\
\text{where } w^t & = & & \{w_j\}_{j=0}^{j=t},\ \widehat{\beta}_{m,n} = (1-(1-\beta)\frac{m}{n}),\ 0 \le m \le n = 1, 2... \\
& & \Rightarrow\ & \Psi(\pi(PE_{m,n}), a) = \overline{\Psi}(\pi(PE_{m,n}), a) = \pi(PE_{m,n})\ \forall\ a \in A \\
& & \Rightarrow\ & l(\pi(PE_{m,n})) = \pi(PE_2)
\end{aligned}$$

$$\begin{aligned}
\pi(PE_1) & \Rightarrow & & u(\pi(PE_1), w^t) = w_0 + \sum_{i=1}^{t}\delta^i w_i \text{ where } w^t = \{w_j\}_{j=0}^{j=t} \\
& \Rightarrow & & \Psi(\pi(PE_1), a) = \overline{\Psi}(\pi(PE_1), a) = \pi(PE_2)\ \forall\ a \in A \\
& \Rightarrow & & l(\pi(PE_1)) = \pi(PN_{m,n})
\end{aligned}$$

$$\begin{aligned}
\pi(PE_2) &\Rightarrow u(\pi(PE_2), w^t) = w_0 + \sum_{i=1}^{t} \delta^i w_i \text{ where } w^t = \{w_j\}_{j=0}^{j=t} \\
&\Rightarrow \Psi(\pi(PE_2), a) = \overline{\Psi}(\pi(PE_2), a) = \pi(PE_2) \ \forall\ a \in A \\
&\Rightarrow l(\pi(PE_2)) = \pi(PE_{m,n})
\end{aligned}$$

$$\begin{aligned}
\pi(\overline{PE}) &\Rightarrow u(\pi(\overline{PE}), w^t) = w_0 + \sum_{i=1}^{t} \delta^i w_i \text{ where } w^t = \{w_j\}_{j=0}^{j=t} \\
&\Rightarrow \Psi(\pi(\overline{PE}), a) = \overline{\Psi}(\pi(\overline{PE}), a) = \pi(\overline{PE}) \ \forall\ a \in A \\
&\Rightarrow l(\pi(\overline{PE})) = \pi(PN_{m,n})
\end{aligned}$$

The agent with state $\pi(PN_{m,n})$ is partially naive in the sense that he believes that he will have have a self-control problem $\widehat{\beta}$ in all the future periods. This is represented as he thinks he will move to another state $\pi(PE_{m,n})$ that implies a self control problem $\widehat{\beta}$ and he will stay at this state for any taken action[3]. He also thinks that his opponent is an exponential agent, $l(\pi(PN_{m,n})) = \pi(PE_1)$ and he thinks his opponent shares the same beliefs with himself, $l(\pi(PE_1)) = \pi(PN_{m,n})$. Moreover, after he moves to the state $\pi(PE_{m,n})$, his opponent will move to another state too, $\pi(PE_2)$, at which he shares again the same beliefs with his opponent, $l(\pi(PE_{m,n})) = \pi(PE_2)$ and $l(\pi(PE_2)) = \pi(PE_{m,n})$. The agent with state $\pi(\overline{PE})$ is an exponential agent who knows his opponent is in the partially naive state, $l(\pi(\overline{PE})) = \pi(PN_{m,n})$.

The equilibrium concept that we defined in section 2 would represent the *"Naive Backwards Induction" (NBI)* if the game is played between the naive type with state $\pi(N)$ (or partially naive with state $\pi(PN_{m,n})$) and the exponential type with state $\pi(\overline{E})$ (or $\pi(\overline{PE})$). Now we will give an example of how this is represented in our solution concept. Take the example game with one player having one and the other having two decision nodes with a small difference.The only difference is that the outcome w_3 is earned not on date 2 but on date 1. The first player has initial state $\pi_{x_0}(1) = \pi(N)$ and the second player has initial state $\pi(\overline{E})$. We suppose that the outcomes and the discount factors are such that:

$$\begin{aligned}
w_{11} &< \beta\delta w_{21} \\
\delta w_{42} &< w_{22} < w_{32} \\
\beta\delta w_{41} &< w_{31} < \delta w_{41}
\end{aligned}$$

We now apply the equilibrium concept. Player 1 will choose the associated action with $(z^{x_0})^*$ given by this.

$$(z^{x_0})^* = \arg\max\{\ u(\pi_{x_0}(1), z_1),\ u(\pi_{x_0}(1), \widehat{z})\}$$

---

[3] Specifying m and n in a certain way leads the partially naive state to become either completely naive or sophisticated states defined earlier. The case where $m = 0$ refers to the naive case where $\pi(PE_{m,n}) = \pi(PE_2)$. The case where $m = n$ refers to the sophisticated case where $\pi(PN_{m,n}) = \pi(PE_{m,n})$ and $\pi(PE_1) = \pi(PE_2) = \pi(\overline{PE})$.

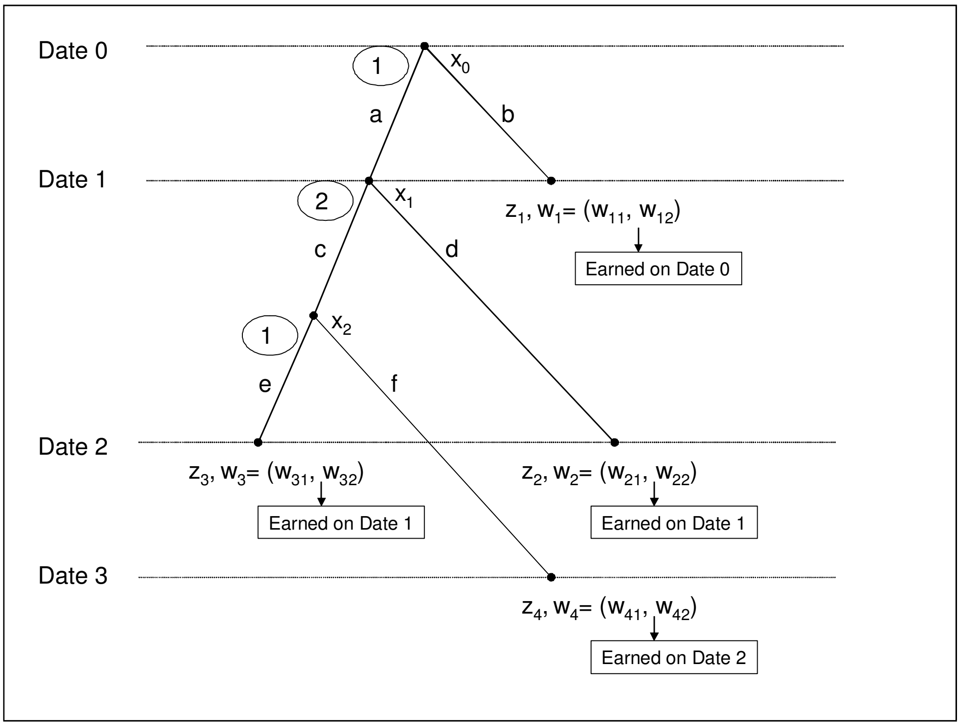
Date 0
1
x0
a
b
Date 1
2
x1
z1, w1= (w11, w12)
Earned on Date 0
c
d
1
x2
e
f
Date 2
z3, w3= (w31, w32)
Earned on Date 1
z2, w2= (w21, w22)
Earned on Date 1
Date 3
z4, w4= (w41, w42)
Earned on Date 2

Figure 10:

$$\begin{aligned} z^{x_1} \quad = \quad & \arg\max\{u(\Psi(l(\pi_{x_0}(1)), a), z_2), \\ & u(\Psi(l(\pi_{x_0}(1)), a),\ \arg\max\{u(\Psi(l(\Psi((l(\pi_{x_0}(1)), a)), c), z_3),\ u(\Psi(l(\Psi((l(\pi_{x_0}(1)), a)), c), z_4)\})\} \end{aligned}$$

$$z^{x_2} = \arg\max\{u(\Psi(\Psi(\pi_{x_0}(1), a), c), z_3),\ u(\Psi(\Psi(\pi_{x_0}(1), a), c), z_4)\}$$

$$\begin{aligned} If\ z^{x_1} \quad &= \quad z_2,\ then\ \widehat{z} = z^{x_1} \\ If\ z^{x_1} \quad &\neq \quad z_2,\ then\ \widehat{z} = z^{x_2} \end{aligned}$$

We first find z^{x_1}. The following argument results in choosing f at node x_2.

$$\arg\max\{u(\pi(E_2), z_3),\ u(\pi(E_2), z_4)\} = \arg\max\{w_{31},\ \delta w_{41}\} = z_4 \Rightarrow action\ f$$

Then z_2 and z_4 will be compared by agent 2 and since the following is satisfied, $z^{x_1} = z_2$ and action d will be chosen at node x_1 :

$$z^{x_1} = \arg\max\{u(\pi(E_2), z_2),\ u(\pi(E_2), z_4)\})\}$$

$$z^{x_1} = \arg\max\{w_{22},\ \delta w_{42}\} = z_2 \Rightarrow action\ d$$

Then, since $z^{x_1} = z_2$ and $(z^{x_0})^* = \arg\max\{\ u(\pi(N), z_1),\ u(\pi(N), z_2)\} = \arg\max\{\beta\delta w_{11},\ \beta\delta^2 w_{21}\} = z_2$, Player 1 will choose *action* a at node x_0.

What actually happens after *action* a is chosen is that player 2 will choose $(z^{x_1})^*$

$$(z^{x_1})^* = \arg\max\{u(\overline{\Psi}(\pi(\overline{E}), a), z_2),\ u(\overline{\Psi}(\pi(\overline{E}), a),\ \arg\max\{u(\Psi(l(\overline{\Psi}(\pi(\overline{E}), a)), c), z_3),\ u(\Psi(l(\overline{\Psi}(\pi(\overline{E}), a)), c), z_4)\})\}$$

$$(z^{x_1})^* = \arg\max\{u(\pi(\overline{E}), z_2),\ u(\pi(\overline{E}),\ \arg\max\{u(\pi(N), z_3),\ u(\pi(N), z_4)\})\}$$

Since $\Psi(l(\overline{\Psi}(\pi(\overline{E}), a)), c) = \Psi(l(\pi(\overline{E})), c) = \Psi(\pi(N), c) = \pi(N)$. This is true because action c is taken in the same period and does not lead to any change in the state of the first player.

$$(z^{x_1})^* = \arg\max\{u(\pi(\overline{E}), z_2),\ u(\pi(\overline{E}),\ \arg\max\{w_{31}, \beta\delta w_{41}\})\} = \arg\max\{u(\pi(\overline{E}), z_2),\ u(\pi(\overline{E}), z_3)\} = z_3$$

Action c is chosen since $(z^{x_1})^* = z_3$. Afterwards, the action $a^* \in a(x_2)$ will be chosen associated with $(z^{x_2})^*$ where $(z^{x_2})^*$ is defined as

$$(z^{x_2})^* = \arg\max\{u(\overline{\Psi}(\overline{\Psi}(\pi(N), a), c), z_3),\ u(\overline{\Psi}(\overline{\Psi}(\pi(N), a), c), z_4)\} = \arg\max\{u(\pi(N), z_3),\ u(\pi(N), z_4)\}$$

Action e is chosen because $(z^{x_2})^* = z_3$. From the perspective of date 0, for agent 1, strategy $(a, f;\ d)$ seemed to be the optimal strategy but the actual play turned out to be $(a, e;\ c)$.

4 Discussion

In this paper, the equilibrium concept for perfect information extensive form games is defined as a path from the initial node to a terminal node given the initial state of the first mover. When this is done, all the evaluations are made based on the initial state of player 1, the first mover. This is because, without loss of generality, player 1 starts the game and his exogenously given initial state determines his own and his opponent's valuations of each terminal node since the opponent's conjectured state is also determined by his initial state. Since the states embrace, almost, all the information structure of the game, by definition, and player 1 is the first mover, everything is driven by player $1's$ initial state, $\pi_{x_0}(1)$. However, this is just for the first move. After the first move made by the first player, the true transitions lead to a potentially different states for each agent, all the evaluations potentially change and now all evaluations are based on what the second player's true state of mind at the node reached is. However, this is true for the second move only and so on. Thus, there is a dynamic structure in this framework where the set of strategies is the same but the optimal strategies of the players potentially change during the course of the game.

A strategy in general tells a player what to do at each of his information set. The optimal strategy or the equilibrium strategy is, in general, obtained by a best response argument. In this framework, it works in a similar way. Each player has a set of strategies (in a perfect information game like this, it is basically the Cartesian product of the set of actions at each of his nodes.) Then, each player (given his state of mind that implies a state for his opponent, assigns a real number to each terminal node and implies a specific sequence of states corresponding to each node that depends on the actions chosen) determines the optimal action for himself and for his opponent at each node based on his given state. At each node, the optimal strategy may vary for both agents depending on how their states evolve. In this sense, the equilibrium concept is dynamic but also deterministic.

In the existing framework, given the states, the equilibrium boils down to a deterministic payoff maximization. Each player has point conjectures about his own future states (characterized by $\Psi(.,.)$) and their opponent's state (characterized by $l(.)$) but both of these may turn out to be wrong. Instead of this, a probabilistic approach can be introduced in such a way that both of these may vary stochastically (both functions $\Psi(.,.)$ and $l(.)$ assign states probabilistically.)

5 Conclusion

In this paper, I study an equilibrium concept for both one and two player extensive form games. The players are modeled as agents being at potentially different states of minds. The state space is inclusive and comprehensive enough such that it embraces almost all the information structure of the game and the all possible types of the players. By this, the aim is to introduce a general enough framework in order to cover different bounded rationality approaches. Finally, a special case that studies the games played by potentially time-inconsistent agents is presented. This paper studies the equilibrium concept in perfect information, one and two player extensive form games. As extensions, it can be generalized to other classes of extensive form games and different psychological, behavioral and other bounded rationality models can be illustrated in a formal way such as regret, limited memory, limited foresight, present bias in a more general sense and addiction.

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