— İKT 495 - ÖDEV 1 - ÇÖZÜMLER —

1. $x_1, x_2, \dots, x_n$ are claims with $x_1 < x_2 < \dots < x_n$. Total resourse is t.

with $t > \sum_{i=1}^{n} x_i$

Proportional rule : $y_i = \dfrac{x_i}{\sum_{i=1} x_i} \cdot t$

Equal surplus rule : $y_i = x_i + \dfrac{t - \sum x_i}{n}$

(a) $x_1 = \min\{x_1, x_2, \dots, x_n\}$. Then, $x_1 < \dfrac{\sum x_i}{n}$.

$$x_1 + \frac{t - \sum x_i}{n} \overset{?}{>} \frac{x_1}{\sum x_i} t \Rightarrow \cancel{\underbrace{x_1 - \frac{\sum x_i}{n}}_{<0}} \overset{?}{>} \frac{t}{\sum x_i} \cancel{\Big(\underbrace{x_1 - \frac{\sum x_i}{n}}_{<0}\Big)}$$

$\Rightarrow 1 < \dfrac{t}{\sum x_i}$ is true. Then for agent 1 ES > PRO.

(b) $x_n = \text{maks}\{x_1, \dots, x_n\}$. Then $x_n > \dfrac{\sum x_i}{n}$.

$$x_n + \frac{t - \sum x_i}{n} \overset{?}{<} \frac{x_n}{\sum x_i} t \Rightarrow \underbrace{x_n - \frac{\sum x_i}{n}}_{>0} \overset{?}{<} \frac{t}{\sum x_i} \Big(\underbrace{x_n - \frac{\sum x_i}{n}}_{>0}\Big)$$

$\Rightarrow 1 < \dfrac{t}{\sum x_i}$ is true. Then for agent n ES < PRO

(c) $x_k = \dfrac{\sum x_i}{n}$, $k \in \{1, \dots, n\}$.

$$x_k + \frac{t - \sum x_i}{n} \overset{?}{=} \frac{x_k}{\sum x_i} \cdot t \Rightarrow \underbrace{x_k - \frac{\sum x_i}{n}}_{=0} \overset{?}{=} \frac{t}{\sum x_i} \Big(\underbrace{x_k - \frac{\sum x_i}{n}}_{=0}\Big)$$

Then for agent k with $x_k = \dfrac{\sum x_i}{n}$ ES = PRO.

2.

$\curvearrowright$ = 800 million.

(a) $$y^{PRO}_{Ali} = \frac{x_{Ali}}{\sum_{i=1}^{100.000} x_i} \cdot t = \frac{32.000}{1.2 \text{ billion}} \cdot (0{,}8 \text{ billion}) \cong 21.333 \text{ TL}$$

$$y^{PRO}_{Ayşe} = \frac{16.000}{1.2} \cdot 0{,}8 \cong 10.666 \text{ TL} \qquad y^{PRO}_{Ahm} = \frac{20.000}{1.2} \cdot 0{,}8 \cong 13.333$$

$$y^{UL}_{Ali} = x_{Ali} + \frac{t - \sum x_i}{n} = 32.000 - \frac{0.8b - 1.2b}{100000} = 32000 - 4000 = 28000 \text{ TL}$$

$$y^{UL}_{AY} = 16000 - 4000 = 12000 \qquad y^{UL}_{AH} = 20000 - 4000 = 16000$$

$$y^{UG}_{Ali} = \min\left\{x_{Ali}, \frac{0.8 \text{ billion}}{100000}\right\} = \min\{32000, 8000\} = 8000 \text{ TL}$$

$$y^{UG}_{AY} = \min\{16000, 8000\} = 8000 \text{ TL}$$

$$y^{UG}_{AH} = \min\{20000, 8000\} = 8000 \text{ TL}$$

(b) Ali favors UL rule since $y^{UL} > y^{PRO} > y^{UG}$.

(c) $1.2 \text{ billion} + 0{,}5 \cdot (1{,}2 \text{ billion}) = 1.8 \text{ billion TL} = \sum x_i$

We know that $t = 1.8$ billion TL, too.

Then every claimant will get its claim, i.e.,

$$y^{PRO}_{Ali} = y^{UL}_{Ali} = y^{UG}_{Ali} = 32000 + 0{,}5 \cdot 32000 = 48000 \text{ TL}$$

$$y^{PRO}_{AY} = y^{UL}_{AY} = y^{UG}_{AY} = 16000 + 0{,}5 \cdot 16000 = 24000 \text{ TL}$$

$$y^{PRO}_{AH} = y^{UL}_{AH} = y^{UG}_{AH} = 20000 + 0{,}5 \cdot 20000 = 30000 \text{ TL}$$

(d) $$y^{PRO}_{ALi} = \frac{x_{ALi}}{\sum x_i} t = \frac{64.000}{2.4 \text{ billion}} \cdot 1{,}8 \text{ billion} = 48000 \text{ TL}$$

$$y^{PRO}_{Ay} = \frac{32000}{2{,}4} \cdot 1{,}8 = 24000 \text{ TL}, \quad y^{PRO}_{AH} = \frac{40000}{2.4} \cdot 1{,}8 = 30000 \text{ TL}$$

$$y^{UL}_{Ali} = x_{ALi} + \frac{t - \sum x_i}{n} = 64000 + \frac{1.8 \text{ bill.} - 2.4 \text{ bill}}{100000} = 64000 - 6000 = 58000 \text{ TL}$$

$$y^{UL}_{AY} = 32000 - 6000 = 26000 \text{ TL} \qquad y^{UL}_{AH} = 40000 - 6000 = 34000 \text{ TL}$$

$1.8 \text{ billion} - 1.1 \text{ billion} = 0.7 \text{ billion}. \quad 0.7 \text{ bill} / 36.000 \cong 19.445$

Then $y^{UG}_{ALi} = y^{UG}_{AY} = y^{UG}_{AH} = 19445$ TL

3.

ⓐ Classical utiliterion approach: max total utility of agents.
Egalitarian approach: want to equalize the utility of agents.
Nash : max product of utilities.

ⓑ Utiliterian:

$$\mathcal{L} = \sqrt{x_1} + 2\sqrt{x_2} + \lambda(100 - x_1 - x_2)$$

$$\frac{\partial L}{\partial x_1} = \frac{1}{2}x_1^{-1/2} - \lambda = 0 \quad \frac{\partial L}{\partial x_2} = x_2^{-1/2} - \lambda = 0 \Rightarrow \frac{1}{2}x_2^{1/2} = x_1^{1/2} \Rightarrow 4x_1 = x_2$$

$$\Rightarrow x_1 = 20,\ x_2 = 80$$

Egaliterian:

$$U_1(x_1) = \sqrt{x_1} = U_2(x_2) = 2\sqrt{x_2} \Rightarrow x_1 = 4x_2 \Rightarrow x_1 = 80,\ x_2 = 20$$

Nash:

$$\mathcal{L} = x_1^{1/2} \cdot 2x_2^{1/2} + \lambda(100 - x_1 - x_2)$$

$$\frac{\partial L}{\partial x_1} = \frac{1}{\cancel{2}}x_1^{-1/2}\cancel{2}x_2^{1/2} - \lambda = 0,\ \frac{\partial L}{\partial x_2} = x_2^{-1/2}x_1^{1/2} - \lambda = 0$$

$$\Rightarrow x_1 = x_2 = 50$$

ⓒ Utiliterian:

$$\mathcal{L} = x_1^{2/3} + x_2^{1/3} + \lambda(100 - x_1 - x_2)$$

$$\frac{\partial L}{\partial x_1} = \frac{2}{3}x_1^{-1/3} - \lambda = 0 \quad \frac{\partial L}{\partial x_2} = \frac{1}{3}x_2^{-2/3} - \lambda = 0 \Rightarrow 2x_1^{-1/3} = x_2^{-2/3}$$

$$\Rightarrow 8x_2^2 = x_1$$

$$\Rightarrow 8x_2^2 + x_2 - 100 = 0 \Rightarrow x_2 \cong 3{,}5 \quad x_1 \cong 96{,}5$$

Egaliterian:

$$U_1(x) = U_2(100 - x) \Rightarrow x^{2/3} = (100 - x)^{1/3} \Rightarrow x^2 + x - 100 = 0 \quad x = x_1 = ---$$

$$\Rightarrow x = x_1 = 9{,}5,\ x_2 = 100 - x = 90{,}5$$

Nash:

$$\mathcal{L} = x_1^{2/3}x_2^{1/3} + \lambda(100 - x_1 - x_2)$$

$$\frac{\partial L}{\partial x_1} = \frac{2}{3}x_1^{-1/3}x_2^{1/3} - \lambda = 0 \quad \frac{\partial L}{\partial x_2} = \frac{1}{3}x_1^{2/3} \cdot x_2^{-2/3} - \lambda = 0 \Rightarrow 2x_1^{-1/3}x_2^{1/3} = x_1^{2/3}x_2^{-2/3}$$

$$\Rightarrow 2x_2 = x_1 \Rightarrow x_1 = \frac{200}{3} \quad x_2 = \frac{100}{3}$$

4.

$$\begin{array}{ccc} 3 & 2 & 2 \\ \hline c & b & a \\ b & a & d \\ a & d & c \\ d & c & b \end{array} \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

(a) According to plurality voting rule c is the winner.

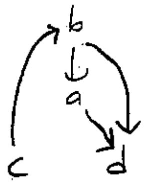
(b) $\overset{2}{a} \prec \overset{5}{b}$, $\overset{4}{a} \succ \overset{3}{c}$, $\overset{7}{a} \succ \overset{0}{d}$, $\overset{2}{b} \prec \overset{5}{c}$, $\overset{5}{b} \succ \overset{2}{d}$, $\overset{3}{c} \prec \overset{4}{d}$

Two cycles: $b \succ a$, $a \succ c$, $c \succ b$

$b \succ d$, $d \succ c$, $c \succ b$.

Condorcet suggest to break the weakest link (repeatedly if necessary)

$c \prec d$ and $a \succ c$ have the weakest links. Then



$\Rightarrow$ $c \succ b \succ a \succ d$.

(c) $a = 13$, $b = 12$, $c = 11$, $d = 6$ Then a is the winner acc. to Borda.

$a \succ b \succ c \succ d$

Remove d.

$$\begin{array}{ccc} 3 & 2 & 2 \\ c & b & a \\ b & a & c \\ a & c & b \end{array} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \begin{array}{l} a = 6 \\ b = 7 \\ c = 8 \end{array}$$

$c \succ b \succ a$. Reversed!

"Independend of irrelevant alternavis" axiom is violated!